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THE OSCILLATIONS OF A SATELLITE PROBE TOWED ON AN INEXTENSIBLE LINE IN AN INHOMOGENEOUS ATMOSPHERE*

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The oscillations of a low-mass satellite towed by an orbital craft or station by means of a long, inextensible line, are considered. The oscillations arise as a result of the action of aderodynamic resistance and tension in the line in an atmosphere of variable density. When the drag coefficient of the satellite-pendulum is known, the period of oscillations is governed by the density of the atmosphere at the flight altitude of the orbital station and the differences in the amplitudes and half-periods in the case of upward and downward deviations from the position of relative equilibrium, are characterized by the density gradient of the atmosphere. Possible ways of utilizing the satellite-pendulum as the means of acquiring information, and the necessary requirements for making the mathematical model more accurate, are discussed.

A satellite in tow launched from an orbital object decelerates under the action of atmospheric resistance and recedes from the orbital object by a distance governed by the length of the line. According to /1/, a typical length of the line is $L = 1-100$ and the mass of the satellite is about 500 kg. The present paper will deal with the motions of a lightweight satellite probe with mass of about 1 kg. When the line unwinds, oscillations occur along it and these must be damped in an appropriate manner. Following this, the aerodynamic resistance and tension in the line cause the satellite probe to enter the mode of steady state oscillations in the atmosphere of variable density. The present paper is concerned with the study of these oscillations.

1. Equations of motion.

Let us consider the relative motion of a satellite probe of mass m attached by means of a weightless inextensible line of constant length L to an orbital craft whose mass is large compared with the mass of the probe. We shall regard the vehicle and the probe as material points. We shall assume that the craft moves along a cricular orbit of radius R_0 with centre at the point O (coinciding with the centre of gravity), at constant angular velocity V_0. We shall introduce a polar coordinate system with polar axis OP. Let the position of the craft be determined by the point K, and that of the probe by the point M. The coordinates of the vehicle and probe in the frame of reference adopted here are (R_0, θ_K) and (R, θ) respectively (Fig.1). We shall also introduce a moving polar coordinate system with polar axis KO, so that the latter frame of reference rotates with angular velocity $\omega_0 = V_0/R_0$. The position of the point M in this system is determined by the distance KM and angle α. We assume that the line is stretched, so that $KM = L$.

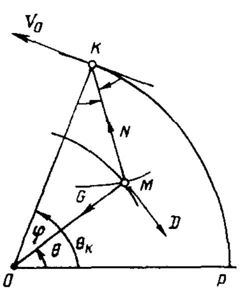

Fig.1

We shall assume for simplicity that there is no wind and the force of resistance (drag) is directed along the tangent to a circle of radius R with centre at the point O. The force of gravity is directed towards the centre, and its magnitude is $G = mg_0 (R_0/R)^2$ where g_0 is the acceleration of free fall in the orbit R_0.

The equation of the oscillations in the rotating polar coordinate system and the expression for the reaction of the line, have the form

$$\alpha^{\cdot\cdot} - Dm^{-1}L^{-1} \sin\beta + g_0L^{-1} (R_0^2R^{-2} \cos\beta - \sin\alpha) = 0 \tag{1.1}$$
$$N = mL (\omega_0 + \alpha^{\cdot})^2 + D \cos\beta + mg_0 (R_0^2R^{-2} \sin\beta - \cos\alpha)$$

(D is the aerodynamic resistance, N is the tension in the line, and β is the angle between the force of gravity and the tangent to the circle of relative motion of the probe).

The swinging probe has a position of equilibrium near $\alpha = {}^1/_2\pi$, i.e. almost on the orbit R_0 behind the craft. The force of aerodynamic resistance generates a momentum which restores the position of equilibrium. The last term in the equation of motion (1.1) characterizes the

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resultant effect of the mutually competing gravity and centrifugal forces, and the term is non-zero even when the acceleration due to gravity is constant.

We shall use the following geometrical relations:

$$\varphi_k = \theta_k - \theta, \quad \beta = {}^1/_2\pi - \alpha - \varphi, \quad \gamma = {}^1/_2\pi - \alpha$$
$$\sin\varphi = LR^{-1}\sin\alpha, \quad \Delta R = R - R_0 = -L\sin\gamma$$

Let us change, in (1.1), from α to γ and use the fact that the length of the line L is small compared with the radius of the orbit R_0. Retaining the principal terms of the expansions in LR_0^{-1}, we obtain

$$\gamma^{\cdot\cdot} + (Dm^{-1}L^{-1} - 3\omega_0^2\cos\gamma)\sin\gamma = 0 \tag{1.2}$$
$$Nm^{-1}L^{-1} = (\omega_0 - \gamma^{\cdot})^2 + 2\omega_0^2\sin^2\gamma - \omega_0^2\cos^2\gamma + Dm^{-1}L^{-1}\cos\gamma$$

System (1.2) is closed by the expression for the aerodynamic resistance D. We shall assume that the density of the atmosphere decreases exponentially with altitude. Since the resistance is proportional to the density, we can write it as follows:

$$D = D_0 e^{\delta\sin\gamma}, \quad D_0 = {}^1/_2 c_D \rho_0 V_0^2 S \tag{1.3}$$

Here D_0 is the aerodynamic resistance at the orbit R_0, the numerical value of the parameter δ depends on the length of the line L and depends only slightly on the flight altitude H, and $\delta \approx 1$ when $L \approx 10$ km.

At the orbit we have $V_0 \approx 8000$ m/sec the density at the flight altitude of $H = 230$ km is $\rho_0 \approx 10^{-10}$ kg/m^3, and the streamlining mode is free-molecular, so that in the case of, for example, a sphere we have $c_0 \approx 2.6$. If the characteristic surface is $S = 1$ m^2, then $D_0 = 10^{-2}$ kg/sec. The corresponding acceleration for a 1 kg probe is 0.01 m/sec^2.

The equation of oscillations (1.1) includes the effect of rotation along the orbit. If the frequency of oscillations ω, governed by the aerodynamic resistance is much greater than the frequency of rotation ω_0, then the second term in the brackets in the equation of oscillations (1.1) can be neglected, and this leads to a pendulum equation in the inertial coordinate system

$$\gamma^{\cdot\cdot} + Dm^{-1}L^{-1}\sin\gamma = 0 \tag{1.4}$$

with

$$\omega^2 = D_0 m^{-1}L^{-1} \gg g_0 R_0^{-1} = \omega_0^2 \tag{1.5}$$

Condition (1.5) imposes a limit on the line length L:

$$L \ll a g_0^{-1} R_0, \quad a = D_0 m^{-1} \tag{1.6}$$

At the altitude $H = 230$ the ratio $a/g_0 = 10^{-3}$ and increases by an order of magnitude when the altitude is reduced to $H = 180$.

2. Linear and weakly non-linear oscillations. We shall first consider the small amplitude oscillations. The main difference between the oscillation Eq. (1.2) and the classical equation of the mathematical pendulum lies in the asymmetrical dependence (1.3) of the resistance D on γ. The asymmetrical property is retained even when we confine ourselves to weakly non-linear, small amplitude oscillations. Expanding the functions in (1.2) in powers of γ and confining ourselves to terms with γ^2, we obtain

$$\gamma^{\cdot\cdot} + (\omega^2 - 3\omega_0^2)\gamma + \delta\omega^2\gamma^2 = 0 \tag{2.1}$$
$$Nm^{-1}L^{-1} = (\omega_0 - \gamma^{\cdot})^2 + 2\omega_0^2\gamma^2 + \omega^2(1 + \delta\gamma - {}^1/_2\gamma^2) - \omega_0^2(1 - {}^1/_2\gamma^2)$$

We note at once the case when the oscillations are linear. When $\delta = 0$, we obtain harmonic oscillations of frequency Ω, with period T_0

$$\Omega^2 = \omega^2 - 3\omega_0^2, \quad T_0 = 2\pi\Omega^{-1} \tag{2.2}$$

If the line is sufficiently short, we can neglect ω_0 in the expression for Ω. For example, when $a = 0.01$ m/sec and $L = 100$ m, we have the period $T_0 \approx 10$ min. Remembering that the period of a single revolution of the satellite about the Earth at an altitude of $H = 200$—300 km is about 90 min, we can find the conditions under which ω_0 should be taken into account.

From the period of oscillations T_0, we can determine the resistance of the body

$$D_0 = mL\,(T_0/(2\pi))^2 \tag{2.3}$$

When the coefficient of resistance c_D is known, formula (2.3) will give the density of the atmosphere ρ_0 at the flight altitude.

Thus we can obtain valuable information from the linear oscillations of a satellite-pendulum on a short line. The information becomes much more extensive when a long line and non-linear oscillations are used.

Let us turn to Eq.(2.1) and solve for it a problem with the following initial conditions:

$$\gamma = \gamma_+, \quad \gamma^{\cdot} = 0 \quad \text{when} \quad t = 0 \tag{2.4}$$

Remembering that $\delta \ll 1$, we shall construct an approximate solution of problem (2.1)-(2.4) using the perturbation method. Let us write

$$\gamma = \gamma_+ Y(t_1), \quad t_1 = \Omega t \tag{2.5}$$

We have the following problem for the function Y:

$$Y^{\cdot\cdot} + Y + \varepsilon Y^2 = 0; \quad Y(0) = 1, \quad Y^{\cdot}(0) = 0 \tag{2.6}$$

$$\varepsilon = \delta\gamma + \omega^2/\Omega^2$$

and we shall seek its solution in the form of expansions in powers of ε

$$Y(t_1, \varepsilon) = Y_0(\tau) + \varepsilon Y_1(\tau) + \varepsilon^2 Y_2(\tau) + \ldots$$

$$t_1 = \Omega t = \tau + \varepsilon f_1(\tau) + \varepsilon^2 f_2(\tau) + \ldots$$

In the zeroth approximation in ε we have

$$Y_0(\tau) = \cos\tau$$

To a first approximation we can write (a prime denotes a derivative with respect to τ)

$$f_1 = 0, \quad Y_1'' + Y_1 = -\cos^2\tau$$

The solution of a differential equation satisfying the homogeneous initial conditions is:

$$Y_1 = -{}^1/_2 + {}^1/_3\cos\tau + {}^1/_6\cos 2\tau$$

The solution to a first approximation has the same period T_0 as the zeroth approximation Y_0. However, this solution is already asymmetrical. Indeed, $Y_1(0) = 0$, $Y_1(\pi) = -{}^2/_3$, $Y_1({}^1/_2\pi) = -{}^2/_3$.

To a second approximation we have $f_2(\tau) = {}^5/_{12}\tau$ and the corresponding solution is

$$Y_2 = -{}^1/_3 + {}^{29}/_{144}\cos\tau + {}^1/_9\cos 2\tau + {}^1/_{48}\cos 3\tau$$

Thus the period becomes distorted only in the second approximation, while the difference in the amplitudes due to the upward and downward deviations becomes apparent already in the first approximation. In the second approximation we have the following relation for the above difference:

$$\Delta Y = Y(0) + Y(\pi) = -{}^2/_3\varepsilon - ({}^2/_3\varepsilon)^2 \tag{2.7}$$

Let us denote by γ_- the maximum deviation of the pendulum during its upward motion, and by $\gamma_+\Delta$ the modulus of the amplitude difference. Since $\gamma_- < 0$, we have $\gamma_+\Delta = -(\gamma_- + \gamma_+)$. Then, according to (2.7) we obtain

$$\gamma_+\Delta = {}^2/_3\varepsilon\gamma_+(1 + {}^2/_3\varepsilon) \tag{2.8}$$

It will be shown below that formula (2.8) holds for amplitudes γ_+ significantly larger than would be expected.

We note that formula (2.8) can be derived more simply by using the energy integral. We shall use this method below for the general non-linear case.

3. Non-linear oscillations. We shall write the complete equation of non-linear oscillations (1.2), taking (1.3) into account, in the form

$$\gamma^{\cdot\cdot} + (\omega^2 e^{\delta\sin\gamma} - 3\omega_0^2\cos\gamma)\sin\gamma = 0 \tag{3.1}$$

$$\omega^2 = D_0 m^{-1} L^{-1}, \quad \omega_0^2 = g_0 R_0^{-1} = V_0^2 R_0^{-2}$$

This equation admits of an energy integral. If $\gamma^{\cdot} = 0$ and $\gamma = \gamma_t$ when $t = 0$, then multiplying (3.1) by $\gamma^{\cdot}$ and integrating with respect to γ from γ_+ to γ, we obtain

$$\gamma^{\cdot 2} + \int_{\gamma_+}^{\gamma} F\,d\gamma = 0 \tag{3.2}$$

$$F = (\omega^2 e^{\delta\sin\gamma} - 3\omega_0^2\cos\gamma)\sin\gamma$$

The integral corresponding to the second term in the integrand can be found by elementary methods; however, since it does not give any advantage when numerical integration is carried out, we shall not separate it out.

If γ reaches its maximum value γ_- when the satellite probe deviates upwards, then $\gamma^{\cdot} = 0$ and the integral in (3.2) vanishes. This condition determines the amplitude γ_- as a function of γ_+, and of the parameters δ and $p = \omega_0^2/\omega^2$.

We can represent the integral in (3.2) in the form of an expansion in powers of δ. Restricting ourselves to the linear approximation in δ, we obtain

$$-\cos\gamma_- + \cos\gamma_+ + {}^1/_2\delta(\gamma_- - \gamma_+) - {}^1/_4\delta(\sin 2\gamma_- - \sin 2\gamma_+) + {}^3/_2 p(\cos^2\gamma_- - \cos^2\gamma_+) = 0 \tag{3.3}$$

Let us write $\gamma_- = -\gamma_+(1+\Delta)$, substitute γ_- into (3.3) and linearize it with respect to Δ. We obtain

$$\gamma_+\Delta = \frac{\gamma_+ - \sin\gamma_+\cos\gamma_+}{(\omega^2 - 3\omega_0^2\cos\gamma_+)\sin\gamma_+}\,\delta \tag{3.4}$$

When $\gamma_+ \ll 1$, the formula transforms into a linear version of (2.8). The process of computing the subsequent terms of the expansion of Δ in a series in δ can obviously be continued, but the calculations are quite bulky and therefore not given here.

Let us turn our attention to the results of integrating the equation (3.2) numerically. The solution depends on two parameters, δ and p. The presence of a difference in (3.1) within the brackets preceding $\sin\gamma$, restricts the range of parameters for which the oscillations relative to $\gamma = 0$ are stable. When $\delta = 0$, i.e. in the case when the atmosphere is homogeneous, the expression within the brackets becomes automatically non-negative when

$$\omega^2 > 3\omega_0^2 \tag{3.5}$$

i.e., when $p < {}^1/_3$.

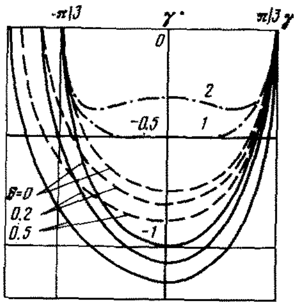

Fig.2

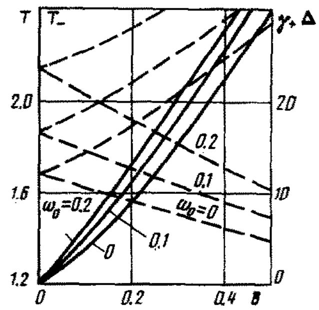

Fig.3

However, when the atmosphere is not homogeneous, i.e., when $\delta > 0$, the resistance D decreases, by virtue of (1.3), with γ, and the expression within the brackets may vanish even when the condition (3.5) holds. Estimates show that when $\delta < 0.5$, it is sufficient to replace condition (3.5) by a slightly stronger condition $\omega^2 > 4\omega_0^2$, i.e. $p < {}^1/_4$. In this case the oscillations relative to $\gamma = 0$ will be stable.

Figures 2 and 3 show the results of solving numerically the integrodifferential Eq.(3.2) which was reduced to the form

$$t = \int_{\gamma_+}^{\gamma} \frac{d\gamma}{\sqrt{2I}}, \quad I = -\int_{\gamma_+}^{\gamma} F\,d\gamma \tag{3.6}$$

The integral I was **evaluated** using Simpson's rule. The first integral was determined using the same method, with a non-uniform step in γ except for the first and last interval in γ. When $\gamma \to \gamma_+$, the integral I is replaced by the expression $I = -F(\gamma_+)(\gamma - \gamma_+)$ which was substituted into (3.6) and integrated, so that $t = 2\sqrt{\gamma_+ - \gamma}/\sqrt{2F(\gamma_+)}$ as $t \to 0$. The same method was used to the case when $\gamma \to \gamma_-$.

When carrying out the numerical integration, we assumed that $\omega = 1$. This condition implies a change to dimensionless time ωt, and the solution will, in this case, depend on the parameters δ and ω_0.

Figure 2 shows the phase trajectories of the family of oscillations for the initial deviation $\gamma_+ = \pi/3$ when $\omega_0 = 0$ and $\delta = 0, 0.2, 0.5$ (solid lines), when $\omega_0 = 0.2$ and $\delta = 0, 0.2, 0.5$ (the dashed lines), and when $\delta = 0$ and $\omega_0 = {}^1/_3; 0.4$ (the dot-dash lines 1 and 2). We can see a clear dependence on the atmosphere inhomogeneity parameter δ reflected in the lack of symmetry of the trajectories relative to $\gamma = 0$. Taking into account the transferred rotational motion will distort the phase trajectory and, in accordance with (2.2), will increase the period of the oscillations.

The difference in the amplitudes as a function of δ is shown in Fig.3 by the solid lines. It is interesting to note that the difference $\gamma_+\Delta$ can be approximated quite well by formula (2.8), so that the family of curves for different ω_0 can be approximately reduced to a single curve. The dependence of the difference of the amplitudes on the homogeneity parameter δ, enables us to determine the index of inhomogeneity of the density of the atmosphere from the measured values of $\gamma_+\Delta$.

The inhomogeneity of the atmosphere in which the oscillations take place leads to the fact that the lower part of the trajectory is traversed by the pendulum faster, and the upper part more slowly than in the case when the atmosphere is homogeneous. Figure 3 shows the dependence of the oscillation half-periods T_+ and T_- for the downward and upward deviations, on the inhomogeneity parameter δ (the dashed lines). Using these relations, or simply the dependence of the difference $\Delta T = T_- - T_+$ on δ, which differs little from the direct proportionality and depends weakly on ω_0, we can also determine δ using the measured value of the difference ΔT.

In all the motions discussed above, the reaction N of the line becomes equal to zero. In general, the oscillations are not planar.

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THE SOLUTIONS OF THE EQUATIONS OF MOTION OF THE KOVALEVSKAYA TOP IN FINITE FORM*

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Elementary transformations of phase variables are used to obtain several novel forms of the system of Euler-Poisson (EP) equations with Kovalevskaya conditions /1/. It is shown that the use of such equations makes possible not only the detection, but also the construction in a finite explicit form, of a solution for all four classes of degenerate motions mentioned by Appel'rot in /4/, and inadequately studied up to now, without using Kovalevskaya quadratures /2, 3/. In particular, an explicit solution is given in a novel form for the third class. The new forms of the equations of motion are used in a unique manner to study some particular results of investigation of degenerate solutions obtained by various methods /5-8/.

1. The initial equations. Using the Kovalevskaya conditions, we will write the EP equations and their algebraic first integrals in the form

$$2p^{\cdot} = qr, \quad 2q^{\cdot} = -rp - c_0\gamma'', \quad r^{\cdot} = c_0\gamma' \tag{1.1}$$

$$\gamma^{\cdot} = r\gamma' - q\gamma'', \quad \gamma'^{\cdot} = p\gamma'' - r\gamma, \quad \gamma''^{\cdot} = q\gamma - p\gamma'$$

$$2(p^2 + q^2) + r^2 - 2c_0\gamma = 6l_1, \quad 2(p\gamma + q\gamma') + r\gamma'' = 2l \tag{1.2}$$

$$\gamma^2 + \gamma'^2 + \gamma''^2 = 1, \quad (p^2 - q^2 + c_0\gamma')^2 + (2pq + c_0\gamma')^2 = k^2$$

where a dot denotes the time derivative. Let us introduce the complex variables

$$x_n = p + \varepsilon_n iq, \quad \xi_n = (p + \varepsilon_n iq)^2 + c_0(\gamma + \varepsilon_n i\gamma'), \quad n = 1, 2 \tag{1.3}$$

$$i = \sqrt{-1}, \quad \varepsilon_1 = 1, \quad \varepsilon_2 = -1$$

and rewrite (1.1) and (1.2) in the form

$$2\varepsilon_n ix_n^{\cdot} = rx_n + c_0\gamma'', \quad 2ir^{\cdot} = x_2^2 - x_1^2 + \xi_1 - \xi_2 \tag{1.4}$$

$$\varepsilon_n i\xi_n^{\cdot} = r\xi_n, \quad 2i\gamma''^{\cdot} = \xi_2 x_1 - \xi_1 x_2 + x_1x_2(x_1 - x_2)$$

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